First, extend the definitions of g_t and f_t for $t<0$: reverse SLE flow.

Lemma. (t and $-t$ change roles of f and g.)

$z \to g_{-t}(z)$ has the same distribution as

$z \to \hat{f}_t(z) - B(\kappa t)$

Proof. Fix $t_1 \in \mathbb{R}$. Let $\xi(t) := B(\kappa t)$

$\tilde{\xi}(t) := \xi(t_1 + t) - \xi(t_1) \overset{\text{Law}}{=} \xi(t)$

$\tilde{g}_t(z) := g_{t_1+t} \circ f_{t_1}(z + \xi(t_1)) - \xi(t_1)$

$\tilde{g}_0(z) = z$, $\quad \underbrace{\tilde{g}_{-t_1}(z) = \hat{f}_{t_1}(z) - \xi(t_1)}$

So $\partial_t \tilde{g}_t(z) = \dfrac{2}{\tilde{g}_t(z) + \xi(t_1) - \xi(t+t_1)} = \dfrac{2}{\tilde{g}_t(z) - \tilde{\xi}(t)}$ $-SLE_\kappa$.

Take $t = -t_1$ ∎

Remark. $\operatorname{Im} g_t(z)$ is decreasing in t $\forall z \in \mathbb{H}$,

$t < T(z) = \sup\{t : z \in f_t(\mathbb{H})\}$.

Time-change: $T_u = T_u(z)$: $\log \operatorname{Im} g_{T_u}(z) = u$

Well-defined: Need to check that $T_u < \infty$, i.e.

$$\lim_{t \to T(z)} \operatorname{Im} g_t(z) = 0,$$

$$\lim_{t \to -\infty} \operatorname{Im} g_t(z) = \infty$$

Proof (of well-defined):

$|\partial_t g_t(z)| = 2|g_t(z) - \xi(t)|^{-1}$

Let $\bar{\xi}(t) = \sup_{0 \le s \le t} |\xi(t)| = \sup_{0 \le s \le t} |B(\kappa t)|$.

Observe: if $|g_s(z)| > \bar{\xi}(t)$, then

$$\partial_s |g_s(z)| \le \frac{2}{|g_s(z)| - \bar{\xi}(t)}$$

Let $U := \{ s \le t : |g_s(z)| > \bar\xi(t) \}$.

$a_s := |g_s(z)| - \bar\xi(t)$. On U, $(a_s^2)' = 2 a_s a_s' \le 4$

By integrating over components of U,
we get $|g_t(z)| = a_t + \bar\xi(t) \le \bar\xi(t) + 2\sqrt{|t|} + |z|$.

Let now $y_t := \operatorname{Im} g_t(z)$. Then $-\frac{\partial}{\partial t} \log y_t = \dfrac{\operatorname{Im} \frac{2}{g_t(z) - \xi(t)}}{\operatorname{Im} g_t(z)} \ge \dfrac{2}{(|t| + \bar\xi(t) + 2\sqrt{|t|})}$

$\bar\xi(t) = \max\limits_{|s| \le |t|} |B(\kappa s)|$

By Khinchin's law of iterated logarithm,

$\bar\xi(t) = \bar{o}(\sqrt{t})$.

So $\frac{\partial}{\partial t} \log y_t \sim \frac{1}{\sqrt{|t|}}$ as $t \to \pm\infty$, so it is not integrable over $(-\infty, 0]$ or $[0, +\infty)$ ∎

$u(t) := \log \operatorname{Im} g_t(z)$

$\partial_t u = \dfrac{-2}{|g_t(z) - \xi(t)|^2}$

inverse time change, time u is z-dependent.

Fix $\hat z = \hat x + i\hat y \in \mathbb{H}$. $\forall u \in \mathbb{R}$, let $z(u) := g_{T_u(\hat z)}(\hat z) - \xi(T_u(\hat z))$

$X(u) := \operatorname{Re} z(u)$

$y(u) = \operatorname{Im} z(u) = \exp u$

$\psi(u) := \dfrac{\hat y}{y(u)} |g'_{T_u}(\hat z)| = \hat y e^{u} |g'_{T_u}(\hat z)|$

Theorem. Let $\nu = -\operatorname{sign}(\log \hat y)$ (i.e. $\nu = 1 \Leftrightarrow \hat y < 1$)

$b \in \mathbb{R}$. $a = 2b + \nu \kappa \frac{b(1-b)}{2}$, $\lambda = 4b + \nu\kappa \frac{b(1-2b)}{2}$.

Set $F(\hat z) = F_b(\hat z) := \hat y^{a} E\left((1 + X(0)^2)^2 |g'_{T_0}(\hat z)|^{a} \right)$.

<u>Then</u> $F(\hat z) = \left(1 + \left(\frac{\hat x}{\hat y} \right)^2 \right)^b \hat y^{\lambda}$

<u>Proof.</u> Note that

$$du = \frac{-2}{|g_t(z) - \xi(t)|^2} dt = -2 |z(u)|^{-2} dt$$

Define a new process:

$$\hat{B}(u) := -\sqrt{\frac{2}{\kappa}} \int_0^{T_u} |Z(u(t))|^{-1} d\xi(t) \qquad \hat{B} \text{ - local martingale}$$

$$d\langle \hat{B}, \hat{B} \rangle_t = -2|Z|^{-2} dt = du$$

So $\hat{B}_u$ is a standard Brownian motion.

Let $\hat{F}(x+iy) := \left(1 + \left(\frac{x}{y}\right)^2\right)^b y^\lambda$

$$M_u := \psi(u)^a \hat{F}(z(u)).$$

By Itô: $\quad dM_u = -2M \frac{bx}{x^2+y^2} d\xi = \sqrt{2\kappa}\, M \frac{bx}{\sqrt{x^2+y^2}} d\hat{B}$

So M_u is a local martingale.

$$\langle M, M \rangle_u = 2\kappa \int_0^u M_s^2 \frac{b^2 x_s^2}{x_s^2 + y_s^2} ds \le c(T) \int_0^u (M_s^2 + 1)\, ds \text{ as long as } u \le T.$$

By a Lemma we had before, M is a martingale.

So $\psi(\hat{u}) \hat{F}(\hat{z}) = E(\psi(0)^a \hat{F}(z(0)))$

$$\hat{y} F(\hat{z}) \quad = \quad E(\psi(\hat{u}) \hat{F}(\hat{z}))$$

Lemma (The Key Estimate) Let $b \in [0, 1 + 4/\kappa]$, $a = 2b + \frac{\kappa b - b}{2}$, $\lambda = 4b + \kappa b(1-2b)/2$.
Then $\exists\, c(\kappa, b)$: $\forall t \in [0,1]$, $y > 0$, $\delta \in [0,1]$, $x \in \mathbb{R}$.

$$P\left(|\hat{f}_t'(x+iy)| > \frac{\delta}{y}\right) \le c(\kappa, b) \left(1 + \frac{x^2}{y^2}\right)^b \left(\frac{y}{\delta}\right)^\lambda A(\delta, a - \lambda)$$

where

$$A(\delta, \mu) := \begin{cases} \delta^{-\mu}, & \mu > 0 \\ 1 + |\log \delta|, & \mu = 0 \\ 1, & \mu < 0 \end{cases}$$

Proof. Assume $\delta > y$, otherwise RHS is $\ge C(\kappa, b)$, and we can take $C(\kappa, b) \ge 1$.

$\hat{f}_t'(z)$ has the same distribution as $g_{-t}'(z)$.

Denote: $u_1 := \log(\operatorname{Im} g_{-t}(x+iy))$

Observe: $\left|\partial_u \log|g_t'(z)|\right| = \dfrac{\mathrm{Re}\left((g_t(z)-\xi(t))^2\right)}{|g_t(z)-\xi(t)|^2} \le 1$

So for any u, $\left|\dfrac{g_{-t}'(z)}{g_{T_u}'(z)}\right| \le \exp(|u-u_1|)$.

So $P\left(|g_{-t}'(z)| \gtrsim \frac{\delta}{y}\right) \lesssim \sum_{j=\lceil \log y \rceil}^{0} P\left(|g_{T_j}'(z)| \gtrsim \frac{\delta}{y}\right)$

All the constants in "$\gtrsim$" and "$\lesssim$" are independent on y and t because $|u_1| = |\log(\mathrm{Im}\, g_{-t}(z))| \le C_1$ for some universal C_1, since $y, t \le 1$ (and thus $\mathrm{Hcap}\; g_{-t} \le 2$)

By Schwarz lemma, $y|g'(z)| \le \mathrm{Im}\, g(z)$ $(g: \mathbb{H} \to \mathbb{H})$

So $P\left(|g_{-t}'(z)| \gtrsim \frac{\delta}{y}\right) \lesssim \sum_{j=\lceil \log \delta \rceil} P\left(|g_{T_j}'(z)| \gtrsim \frac{\delta}{y}\right)$

enough to start at $\lceil \log \delta \rceil$.

Now, we can use scale invariance to get that if $j \in [\log y, 0]$, then

$E\left(y^a e^{-ja} |g_{T_j(z)}'(z)|^a\right) \le F(e^{-j} z)$.

So $P\left(|g_{T_j(z)}'(z)| \ge \frac{\delta}{y}\right) = P\left(|g_{T_j(z)}'(z)|^a y^a \delta^{-a} \ge 1\right) \overset{\text{Chebyshev}}{\le}$

$E\left(|g_T'(z)|^a y^a \delta^{-a}\right) \le \delta^a e^{+ja} F(e^{-j} z)$.

So $P\left(|g_{-t}'(z)| \gtrsim \frac{y}{\delta}\right) \lesssim \left(1+\frac{x^2}{y^2}\right)^b \delta^{-a} y^\lambda \sum_{j=\lceil \log \delta \rceil}^{0} e^{j(a-\lambda)}$

Which gives the lemma.